Appendix A

# Complements on Gaussian random fields

## Contents



"I have had my results for a long time: but I do not yet know how I am to arrive at them."
— Carl Friedrich Gauß
Quoted in Arber (1954), *The Mind and the Eye*


## Abstract

This appendix provides complements on Gaussian random fields. It offers a mathematical exposition of their definition and demonstrates well-known properties, used in particular in chapter 1 and for the generation of initial conditions for cosmological simulations (section B.6).


## A.1 Characteristic function

**Definition A.1.** For a random scalar vector $\lambda \in \mathbb{C}^n$ whose pdf is $\mathcal{P}(\lambda)$, the characteristic function $\varphi_\lambda$ is defined as the inverse Fourier transform of $\mathcal{P}(\lambda)$. In other words, it is the expectation value of $\mathrm{e}^{\mathrm{i}t^*\lambda}$, where $t \in \mathbb{C}^n$ is the argument of the characteristic function (e.g. Manolakis, Ingle & Kogon, 2000):

$$\varphi_\lambda(t) \equiv \left\langle \mathrm{e}^{\mathrm{i}t^*\lambda} \right\rangle = \int_{\mathbb{C}} \mathrm{e}^{\mathrm{i}t^*\lambda}\, \mathcal{P}(\lambda)\, \mathrm{d}\lambda. \tag{A.1}$$

Characteristic functions have well-known properties. In particular, an important theorem is the following.

**Theorem A.2. (Kac's theorem).** Let $\lambda_1, \lambda_2 \in \mathbb{C}^n$ be random vectors. The following statements are equivalent:

1. $\lambda_1$ and $\lambda_2$ are independent (we note $\lambda_1 \perp\!\!\!\perp \lambda_2$),
2. the characteristic function of the joint random vector $(\lambda_1, \lambda_2)$ is the product of the characteristic functions of $\lambda_1$ and $\lambda_2$ i.e. $\varphi_{(\lambda_1,\lambda_2)} = \varphi_{\lambda_1}\varphi_{\lambda_2}$.

*Proof.* 1. $\Rightarrow$ 2. is straightforward using $\langle f(\lambda_1) g(\lambda_2)\rangle = \langle f(\lambda_1)\rangle \langle g(\lambda_2)\rangle$.

2. $\Rightarrow$ 1. Let $\widetilde{\lambda_1}$ and $\widetilde{\lambda_2}$ be random vectors such that $\widetilde{\lambda_1}$ and $\lambda_1$ have the same pdf, $\widetilde{\lambda_2}$ and $\lambda_2$ have the same pdf and $\widetilde{\lambda_1} \perp\!\!\!\perp \widetilde{\lambda_2}$. Then

$$\begin{aligned} \varphi_{(\lambda_1,\lambda_2)} &= \varphi_{\lambda_1}\varphi_{\lambda_2} && \text{using 2.} \\ &= \varphi_{\widetilde{\lambda_1}}\varphi_{\widetilde{\lambda_2}} && \text{using the pdfs} \\ &= \varphi_{(\widetilde{\lambda_1},\widetilde{\lambda_2})} && \text{using 1.} \Rightarrow \text{2.} \end{aligned}$$

i.e. the characteristic functions of $(\lambda_1, \lambda_2)$ and $(\widetilde{\lambda_1}, \widetilde{\lambda_2})$ coincide. From the uniqueness of the inverse Fourier transform we conclude that $(\lambda_1, \lambda_2)$ and $(\widetilde{\lambda_1}, \widetilde{\lambda_2})$ are drawn from the same distribution, hence $\lambda_1 \perp\!\!\!\perp \lambda_2$. $\square$

## A.2 General definition of a Gaussian random vector

**Definition A.3.** A multivariate random scalar vector $\lambda \in \mathbb{C}^n$ is a Gaussian random vector if and only if there exists a vector $\mu \in \mathbb{C}^n$ and a Hermitian, positive semi-definite matrix $C \in \mathcal{M}_n(\mathbb{C})$ such that the characteristic function of $\lambda$ is

$$\varphi_\lambda(t) = \exp\left(\mathrm{i}t^*\mu - \frac{1}{2}t^*Ct\right). \tag{A.2}$$

In this case, $\mu$ and $C$ are called the mean and covariance matrix of $\lambda$, respectively, and we note $\lambda \sim \mathcal{N}_n\left[\mu, C\right]$. Here, the covariance matrix is allowed to be singular. This definition generalizes the one given in section 1.2.3.1, as we see from the following theorem.

**Theorem A.4.** When $C$ is positive-definite (and therefore invertible), the distribution of $\lambda$ has a multivariate normal density

$$\mathcal{P}(\lambda|\mu, C) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}(\lambda-\mu)^* C^{-1}(\lambda-\mu)\right). \tag{A.3}$$

*Proof.* By explicitly computing the inverse Fourier transform of the multivariate normal distribution above (i.e. calculating the Gaussian integral), we can check that the characteristic function of this distribution coincides with the value of equation (A.2). From the uniqueness of the inverse Fourier transform, we conclude that $\lambda$ is drawn from the distribution whose pdf is given above. □

When this condition is fulfilled, we say that $\lambda$ is *non-degenerate*.

## A.3 Some well-known properties of Gaussian random vectors

**Proposition A.5.** Linear transformations preserve Gaussianity, i.e. for all $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $b \in \mathbb{C}^m$, if $\lambda \sim \mathcal{N}_n\left[\mu, C\right]$, then $A\lambda + b \sim \mathcal{N}_m\left[A\mu + b, ACA^*\right]$.

*Proof.* The characteristic function of $A\lambda + b$ is, for all $s \in \mathbb{C}^m$,

$$\begin{aligned}
\varphi_{A\lambda+b}(s) &= \left\langle \mathrm{e}^{\mathrm{i}s^*(A\lambda+b)} \right\rangle \\
&= \left\langle \mathrm{e}^{\mathrm{i}(A^*s)^*\lambda} \right\rangle \mathrm{e}^{\mathrm{i}s^*b} \\
&= \varphi_\lambda(A^*s)\,\mathrm{e}^{\mathrm{i}s^*b} \\
&= \exp\left(\mathrm{i}(A^*s)^*\mu - \frac{1}{2}(A^*s)^*CA^*s\right)\exp\left(\mathrm{i}\,s^*b\right) \\
&= \exp\left(\mathrm{i}s^*(A\mu+b) - \frac{1}{2}s^*(ACA^*)s\right).
\end{aligned}$$

□

**Proposition A.6.** Adding two independent Gaussians yields a Gaussian, i.e. if $\lambda_1 \sim \mathcal{N}_n\left[\mu_1, C_1\right]$, $\lambda_2 \sim \mathcal{N}_n\left[\mu_2, C_2\right]$ and $\lambda_1 \perp\!\!\!\perp \lambda_2$, then $\lambda_1 + \lambda_2 \sim \mathcal{N}_n\left[\mu_1+\mu_2, C_1+C_2\right]$.

*Proof.* The independence of $\lambda_1$ and $\lambda_2$ implies the independence of $\mathrm{e}^{\mathrm{i}t^*\lambda_1}$ and $\mathrm{e}^{\mathrm{i}t^*\lambda_2}$. Therefore,

$$\varphi_{\lambda_1+\lambda_2}(t) = \left\langle \mathrm{e}^{\mathrm{i}t^*(\lambda_1+\lambda_2)} \right\rangle = \left\langle \mathrm{e}^{\mathrm{i}t^*\lambda_1}\mathrm{e}^{\mathrm{i}t^*\lambda_2} \right\rangle = \left\langle \mathrm{e}^{\mathrm{i}t^*\lambda_1} \right\rangle \left\langle \mathrm{e}^{\mathrm{i}t^*\lambda_2} \right\rangle = \varphi_{\lambda_1}(t)\varphi_{\lambda_2}(t).$$

Using the characteristic functions of $\lambda_1$ and $\lambda_2$ yields

$$\varphi_{\lambda_1+\lambda_2}(t) = \exp\left(\mathrm{i}t^*\mu_1 - \frac{1}{2}t^*C_1t\right)\exp\left(\mathrm{i}t^*\mu_2 - \frac{1}{2}t^*C_2t\right) = \exp\left(\mathrm{i}t^*(\mu_1+\mu_2) - \frac{1}{2}t^*(C_1+C_2)t\right).$$

□

## A.4 Marginal and conditionals of Gaussian random vectors

To study the partition of Gaussian random vectors, let us define

$$\lambda = \begin{pmatrix}\lambda_x \\ \lambda_y\end{pmatrix}, \quad \mu = \begin{pmatrix}\mu_x \\ \mu_y\end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix}C_{xx} & C_{xy} \\ C_{yx} & C_{yy}\end{pmatrix}, \tag{A.4}$$

where $\lambda_x, \mu_x \in \mathbb{C}^m$, $C_{xx} \in \mathcal{M}_m(\mathbb{C})$, $\lambda_y, \mu_y \in \mathbb{C}^{n-m}$, $C_{yy} \in \mathcal{M}_{n-m}(\mathbb{C})$, $C_{xy} \in \mathcal{M}_{m\times(n-m)}(\mathbb{C})$ and $C_{yx} = (C_{xy})^* \in \mathcal{M}_{(n-m)\times m}(\mathbb{C})$. We assume that $m < n$ and we want to prove that the marginal and conditional distributions of $\lambda_x$ and $\lambda_y$ are Gaussians with parameters given by equations (1.19)–(1.22) and (1.23)–(1.26). By symmetry, we limit the discussion to $\lambda_x$ and $\lambda_x|\lambda_y$.

**Proposition A.7.** The marginal distribution of $\lambda_x$ is that of a Gaussian random vector with mean $\mu_x$ and variance $C_{xx}$.

*Proof.* Consider $A = \begin{pmatrix}\mathbf{1}_{xx} & \mathbf{0}_{xy} \\ \mathbf{0}_{yx} & \mathbf{0}_{yy}\end{pmatrix}$. Proposition A.5. yields $A\lambda = \lambda_x \sim \mathcal{N}_m\left[A\mu, ACA^*\right] = \mathcal{N}_m\left[\mu_x, C_{xx}\right]$. □

Let us now consider the conditionals.

**Lemma A.8.** $\lambda_x$ and $\lambda_y$ are independently distributed if and only if $C_{xy} = \mathbf{0}_{xy}$.

*Proof.* This proposition follows by considering the characteristic function of $\lambda$:

$$\begin{aligned}
\varphi_\lambda(t) &= \varphi_{(\lambda_x,\lambda_y)}(t_x, t_y) \\
&= \exp\left(\mathrm{i}t^*\mu - \frac{1}{2}t^*Ct\right) \\
&= \exp\left(\mathrm{i}t_x^*\mu_x + \mathrm{i}t_y^*\mu_y - \frac{1}{2}t_x^*C_{xx}t_x - \frac{1}{2}t_x^*C_{xy}t_y - \frac{1}{2}t_y^*C_{yy}t_y - \frac{1}{2}t_y^*C_{yx}t_x\right) \\
&= \varphi_{\lambda_x}(t_x)\varphi_{\lambda_y}(t_y)\exp\left(-t_x^*C_{xy}t_y\right)
\end{aligned}$$

and using Kac's theorem (theorem A.2.), $\lambda_x \perp\!\!\!\perp \lambda_y \Leftrightarrow \varphi_{(\lambda_x,\lambda_y)} = \varphi_{\lambda_x}\varphi_{\lambda_y} \Leftrightarrow C_{xy} = \mathbf{0}_{xy}$. □

**Definition A.9.** Let $C_{xx.y} \equiv C_{xx} - C_{xy}C_{yy}^{-1}C_{yx}$, the so-called generalized Schur-complement of $C_{yy}$ in $C$.

**Lemma A.10.**

$$\begin{pmatrix}\lambda_x - C_{xy}C_{yy}^{-1}\lambda_y \\ \lambda_y\end{pmatrix} \sim \mathcal{N}_n\left[\begin{pmatrix}\mu_x - C_{xy}C_{yy}^{-1}\mu_y \\ \mu_y\end{pmatrix}, \begin{pmatrix}C_{xx.y} & \mathbf{0}_{xy} \\ \mathbf{0}_{yx} & C_{yy}\end{pmatrix}\right]. \tag{A.5}$$

*Proof.* Consider $A = \begin{pmatrix}\mathbf{1}_{xx} & -C_{xy}C_{yy}^{-1} \\ \mathbf{0}_{yx} & \mathbf{1}_{yy}\end{pmatrix}$. The lemma follows by considering $A\lambda$ and using proposition A.5. □

**Proposition A.11.** The conditional distribution of $\lambda_x$ given $\lambda_y$ is the Gaussian distribution given by

$$\mathcal{N}_m\left[\mu_x + C_{xy}C_{yy}^{-1}(\lambda_y - \mu_y), C_{xx.y}\right].$$

*Proof.* Since $\lambda_x - C_{xy}C_{yy}^{-1}\lambda_y$ and $\lambda_y$ have zero covariance matrix (lemma A.10.), they are independently distributed according to lemma A.8. Therefore, using also the result obtained for the marginals (proposition A.7.), we get

$$\begin{aligned}
(\lambda_x - C_{xy}C_{yy}^{-1}\lambda_y)|\lambda_y &\sim \lambda_x - C_{xy}C_{yy}^{-1}\lambda_y \\
&\sim \mathcal{N}_m\left[\mu_x - C_{xy}C_{yy}^{-1}\mu_y, C_{xx.y}\right]
\end{aligned}$$

and hence

$$\begin{aligned}
\lambda_x|\lambda_y &\sim (\lambda_x - C_{xy}C_{yy}^{-1}\lambda_y + C_{xy}C_{yy}^{-1}\lambda_y)|\lambda_y \\
&\sim \mathcal{N}_m\left[\mu_x + C_{xy}C_{yy}^{-1}(\lambda_y - \mu_y), C_{xx.y}\right]
\end{aligned}$$

by just translating the above normal density by the constant vector $C_{xy}C_{yy}^{-1}\lambda_y$. □